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VARIATIONAL METHODS AND ALMOST PERIODIC SOLUTIONS OF SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract

By means of variational methods, we study the existence and uniqueness of almost periodic solutions for a class of second order neutral functional differential equations with infinite delay.

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1 Introduction

Neutral functional differential equations (abreviated NFDE) with infinite delay arise in many areas of applied mathematics. For this reason the study of these types of equations has received great attention for several decades. Amoung the wide literature devoted to retarded differential equations, for the infinite delay we refer to [18, 22, 25, 28] and the references therein, and for the second-order equations we refer to [20, 21].

One of the most attractive aspects of the qualitative theory of this type of equations is the existence of almost periodic solutions, due to their significance in physics, biology, and other. The literature devoted to this subject is fundamentally concerned with first order equations (see for example [28]). In a similar way, second order NFDE have recently been considered in the literature by Henriquez-Vasquez [19], and Diagana-Henriquez-Hernandez [16].

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About the existence of almost periodic solutions, we introduce variational methods. At our knowledge the use of variational methods is new in the study of almost periodic solutions of infinite delay-differential equations.

Elsgolc [17] initiated a theory of calculus of variations with a retarded argument. Later Hughes [23] and Sabbagh [30] provided additional results of this type of calculus of variations. There also exists a theory of calculus of variations in mean time developed by J. Blot (cf. [6, 7, 8, 9, 10, 11]) to study the almost periodic solutions for some (non-delayed) differential equations. In their work [3] M. Ayachi and J. Blot extend Shu and Xu [31] and Y. Li [27] variational setting for periodic solutions of nonlinear NFDE (finite delay case) to the almost periodic setting. In the present work we introduce a new formalism of calculus of variations in mean time with infinite delay argument.

The basic problem is the following:

$$\begin{cases} \text{Minimize } \lim\limits_{T\to\infty} \dfrac{1}{2T}\displaystyle\int_{-T}^{T} L(u(t), u_t, \nabla u(t), (\nabla u)_t, t)dt, \\ \\ \text{subject to the constraint } \ u \in B^{1,2}(\mathbb{R}, \mathbb{R}^n). \end{cases} \tag{1.1}$$

We then obtain the existence and uniqueness of weak almost periodic solutions (in the sense of Besicovitch) to the second order NFDE with infinite delay in the following form

$$\begin{cases} D_1 L(u(t), u_t, u'(t), u'_t, t) + \mathcal{T}^* D_2 L(u(t), u_t, u'(t), u'_t, t) \\ \\ \quad = \dfrac{d}{dt}\left[D_3 L(u(t), u_t, u'(t), u'_t, t) + \mathcal{T}^* D_4 L(u(t), u_t, u'(t), u'_t, t)\right], \end{cases} \tag{1.2}$$

where L is a differentiable function, D_j denotes the partial differential with respect to the jth vector variable, $\mathcal{T}^*$ denotes the adjoint of the linear operator $\mathcal{T}$ which will be specified later, and for $t \in \mathbb{R}$, the history function u_t is defined by

$$u_t(\theta) := u(t+\theta), \text{ for } \theta \in (-\infty, 0].$$

Note that a special case of (1.2) is the following forced second order differential equation with infinite delay

$$u''(t) + D_1 G(u(t), u_t) + \mathcal{T}^* D_2 G(u(t), u_t) = e(t).$$

The paper is organized as follows: in section (2) we precise the notations about the function spaces and we recall some definitions and properties that will be used in this work. In section (3) we establish some preliminary results. In section (4) we establish a variational formalism suitable to the Besicovitch almost-periodic solutions and Euler Lagrange equation, and in section (5) we obtain results of existence and uniqueness of almost periodic solutions.

2 Definitions and notations

Let $\mathbb{E}$ be a Banach space with norm $|.|_{\mathbb{E}}$.

For $p \in [1,\infty)$, we denote by $B^p(\mathbb{R},\mathbb{E})$ the space of Besicovitch almost-periodic (Besicovitch a.p. for short) functions from $\mathbb{R}$ in $\mathbb{E}$ (cf. [4, 5, 29]). We recall that $B^p(\mathbb{R},\mathbb{E})$ is the completion of $\mathcal{AP}^0(\mathbb{R},\mathbb{E})$ (in $L^p_{loc}(\mathbb{R},\mathbb{E})$) with respect to the norm

$$\|f\|_{B^p(\mathbb{R},\mathbb{E})} = \mathfrak{M}_t\left\{|f(t)|^p_{\mathbb{E}}\right\}^{\frac{1}{p}} := (\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^p_{\mathbb{E}}\,dt)^{\frac{1}{p}},$$

where $\mathcal{AP}^0(\mathbb{R},\mathbb{E})$ is the space of Bohr almost-periodic (Bohr a.p. for short) functions from $\mathbb{R}$ in $\mathbb{E}$ (cf. [5, 12, 13, 26, 29]).

Recall the following useful fact: if $(f_m)_m$ is a sequence in $\mathcal{AP}^0(\mathbb{R},\mathbb{E})$ and if $f \in L^p_{loc}(\mathbb{R},\mathbb{E})$ satisfies

$$\overline{\mathfrak{M}}_t\left\{|f_m - f|^p_{\mathbb{E}}\right\}^{\frac{1}{p}} := \left(\limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_m - f|^p_{\mathbb{E}}\,dt\right)^{\frac{1}{p}} \to 0 \quad (m\to\infty),$$

then $f \in B^p(\mathbb{R},\mathbb{E})$, and we have $\|f_m - f\|_{B^p(\mathbb{R},\mathbb{E})} \to 0 \quad (m\to\infty)$.

We recall the 'Mean Value Theorem' (c.f. ([4, page 93], [5, page 244-245], [13, page 45])) : If $f \in B^p(\mathbb{R},\mathbb{E})$, the mean value of f exists in $\mathbb{E}$ and satisfies the following property

$$\mathfrak{M}\{f\} = \mathfrak{M}^+\{f\} = \mathfrak{M}^-\{f\}, \tag{2.1}$$

where

$$\begin{cases}\mathfrak{M}^+\{f\} := \lim_{T\to+\infty}\frac{1}{T}\int_0^T f(t)dt\\ \mathfrak{M}^-\{f\} := \lim_{T\to+\infty}\frac{1}{T}\int_{-T}^0 f(t)dt.\end{cases}$$

When $\lambda \in \mathbb{R}$, $a(f;\lambda) := \mathfrak{M}_t\left\{f(t)e^{-i\lambda t}\right\}$ are called the Fourrier-Bohr coefficients of f, and

$$\Lambda(f) := \{\lambda \in \mathbb{R} : a(f,\lambda) \neq 0\}.$$

When $p = 2$, and $(\mathbb{E},(.|.))$ is a Hilbert space with norm associated to the inner product $|.|_{\mathbb{E}} := \sqrt{(.|.)}$, $B^2(\mathbb{R},\mathbb{E})$ is a Hilbert space, and its norm $\|.\|_2$ associated to the inner product $\langle f|g\rangle_{B^2(\mathbb{R},\mathbb{E})} := \mathfrak{M}\{(f|g)\}$, (cf. [4, 29]).

We use the generalized derivative (when it exists), ∇f defined by

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\nabla f(t) - \frac{1}{s}(f(t+s) - f(t))\right|^2_{\mathbb{E}} dt \to 0 \quad (s\to 0),$$

to define

$$B^{1,2}(\mathbb{R},\mathbb{E}) := \left\{f \in B^2(\mathbb{R},\mathbb{E}) : \quad \nabla f \in B^2(\mathbb{R},\mathbb{E})\right\},$$

which is a Hilbert space for the inner product

$$\langle f|g\rangle_{B^{1,2}(\mathbb{R},\mathbb{E})} := \langle f|g\rangle_{B^2(\mathbb{R},\mathbb{E})} + \langle\nabla f|\nabla g\rangle_{B^2(\mathbb{R},\mathbb{E})},$$

and we denote by $\|f\|_{B^{1,2}(\mathbb{R},\mathbb{E})} := \sqrt{\langle f|f\rangle_{B^{1,2}(\mathbb{R},\mathbb{E})}}$, (c.f. [11, 15]).

We recall that $f \in B^2(\mathbb{R},\mathbb{E})$ if and only if there exists a sequence $\{a_\lambda\}_\lambda \in l^2(\mathbb{E})$, such that $f(t) \sim \sum_{\lambda\in\mathbb{R}} a_\lambda e^{i\lambda t}$, and in this case we have

$$f \in B^{1,2}(\mathbb{R},\mathbb{E}) \Leftrightarrow \sum_{\lambda\in\mathbb{R}} \lambda^2\,|a_\lambda|^2_{\mathbb{E}} < +\infty, \tag{2.2}$$

and
$$\nabla f(t) \sim \sum_{\lambda\in\mathbb{R}} i\lambda a_\lambda e^{i\lambda t}. \tag{2.3}$$

If $\mathfrak{X}$ and $\mathfrak{Y}$ are two Banach spaces, $\mathcal{APU}(\mathfrak{X}\times\mathbb{R},\mathfrak{Y})$ stands for the space of functions $F:\mathfrak{X}\times\mathbb{R}\to\mathfrak{Y}$, $(x,t)\mapsto F(x,t)$, which are almost periodic in t uniformly with respect to x in the classical sense given in [32, Chapter 1].

To make the writing less heavy, we sometimes use the following notations :

- $\mathcal{B} := B^2((-\infty,0],\mathbb{R}^n)$, which is a Hilbert space for the norm $|.|_{\mathcal{B}}$ associated to the inner product $(f|g)_{\mathcal{B}} := \mathfrak{M}_\theta^-\{f(\theta).g(\theta)\}$.
- $\mathbb{H} := \mathbb{R}^n\times\mathcal{B}\times\mathbb{R}^n\times\mathcal{B}$ is the product Hilbert endowed by the product norm $|.|_{\mathbb{H}}$, defined by
$$|X|_{\mathbb{H}} := \left(|x_1|^2+|\varphi_1|_{\mathcal{B}}^2+|x_2|^2+|\varphi_2|_{\mathcal{B}}^2\right)^{\frac{1}{2}},$$
for all $X := (x_1,\varphi_1,x_2,\varphi_2)\in\mathbb{H}$.
- $u^- := u_{|(-\infty,0]}$, is the restriction of the function $u:\mathbb{R}\to\mathbb{R}^n$ on $(-\infty,0]$.
- $\underline{u}(t) := (u(t),u_t,\nabla u(t),(\nabla u)_t)$, where $u\in B^{1,2}(\mathbb{R},\mathbb{R}^n)$.
- When $u\in L^2_{loc}(\mathbb{R},\mathbb{R}^n)$ (Lebesgue space), we denote by $\tilde{u}:\mathbb{R}\longrightarrow L^2_{loc}((-\infty,0],\mathbb{R}^n)$ the function defined by $\tilde{u}(t)(\theta) := u(t+\theta)$.

3 Preliminary results

Lemma 3.1. *Suppose $u\in B^2(\mathbb{R},\mathbb{R}^n)$ is such that $u(t)\sim\sum_\lambda a_\lambda e^{i\lambda t}$. Then $u^-\in\mathcal{B}$ and $u^-(\theta)\sim\sum_\lambda a_\lambda e^{i\lambda\theta}$ for all $\theta\in(-\infty,0]$.*

Proof. By using (2.1) we obtain
$$\begin{aligned}\left\|u(.)-\sum_{\lambda=1}^m a_\lambda e^{i\lambda(.)}\right\|^2_{B^2(\mathbb{R},\mathbb{R}^n)} &= \mathfrak{M}_t\left\{\left|u(t)-\sum_{\lambda=1}^m a_\lambda e^{i\lambda t}\right|^2\right\} = \mathfrak{M}_t^-\left\{\left|u(t)-\sum_{\lambda=1}^m a_\lambda e^{i\lambda t}\right|^2\right\}\\ &= \mathfrak{M}_\theta^-\left\{\left|u^-(\theta)-\sum_{\lambda=1}^m a_\lambda e^{i\lambda\theta}\right|^2\right\} = \left|u^-(.)-\sum_{\lambda=1}^m a_\lambda e^{i\lambda(.)}\right|^2_{\mathcal{B}}.\end{aligned}$$

Since $u\in B^2(\mathbb{R},\mathbb{R}^n)$ and $u(t)\sim\sum_\lambda a_\lambda e^{i\lambda t}$, we have
$$\lim_{m\to\infty}\left\|u(.)-\sum_{\lambda=1}^m a_\lambda e^{i\lambda(.)}\right\|_{B^2(\mathbb{R},\mathbb{R}^n)} = 0, \text{ and } \sum_\lambda |a_\lambda|^2<\infty,$$
and so
$$\lim_{m\to\infty}\left|u^-(.)-\sum_{\lambda=1}^m a_\lambda e^{i\lambda(.)}\right|_{\mathcal{B}} = 0, \text{ and } \sum_\lambda |a_\lambda|^2<\infty.$$
Which implies that $u^-\in\mathcal{B}$, and its Fourier-Bohr series is $\sum_\lambda a_\lambda e^{i\lambda\theta}$ for all $\theta\in(-\infty,0]$. $\square$

Remark 3.2. Reciprocally, all function $[\theta \mapsto v(\theta)] \in \mathcal{B}$, such that $v(\theta) \sim \sum_\lambda a_\lambda e^{i\lambda\theta}$, possess a unique extension to $\mathbb{R}$ which will be written also v, and $v(t) \sim \sum_\lambda a_\lambda e^{i\lambda t}$, $(t \in \mathbb{R})$.

Lemma 3.3. *Suppose $u \in B^2(\mathbb{R}, \mathbb{R}^n)$ such that $u(t) \sim \sum_\lambda a_\lambda e^{i\lambda t}$. Then the following properties are satisfied*

(i) $\tilde{u} \in B^2(\mathbb{R}, \mathcal{B})$.

(ii) $\tilde{u}(t) \sim \sum_\lambda b_\lambda e^{i\lambda t}$*, where* $b_\lambda : (-\infty, 0] \to \mathbb{R}^n$ *is defined by*

$$b_\lambda(\theta) := a_\lambda e^{i\lambda\theta}.$$

(iii) $\|\tilde{u}\|_{B^2(\mathbb{R},\mathcal{B})} = \|u\|_{B^2(\mathbb{R},\mathbb{R}^n)}$.

Proof. $u \in B^2(\mathbb{R}, \mathbb{R}^n)$, then by using Lemma (3.1) , $u^- \in \mathcal{B}$, and since $\mathcal{B}$ is stable by translation [4], we deduce that for all t in $\mathbb{R}$, $u_t \in \mathcal{B}$ and

$$u_t(\theta) \sim \sum_\lambda a_\lambda e^{i\lambda(t+\theta)}.$$

And so the function $\tilde{u}$ is well defined in $\mathbb{R}$ to $\mathcal{B}$.

Since the mean value is translation invariant (c.f. [29, page 10]), we have for all $t \in \mathbb{R}$, $\theta \in (-\infty, 0]$

$$\begin{aligned}
\left\| \tilde{u}(.) - \sum_{\lambda=1}^{m} b_\lambda e^{i\lambda(.)} \right\|^2_{B^2(\mathbb{R},\mathcal{B})} &= \mathfrak{M}_t \left\{ \left| \tilde{u}(t) - \sum_{\lambda=1}^{m} b_\lambda e^{i\lambda t} \right|^2_{\mathcal{B}} \right\} \\
&= \mathfrak{M}_t \left\{ \mathfrak{M}^-_\theta \left\{ \left| u(t+\theta) - \sum_{\lambda=1}^{m} a_\lambda e^{i\lambda(t+\theta)} \right|^2 \right\} \right\} \\
&= \mathfrak{M}_t \left\{ \mathfrak{M}^-_\theta \left\{ \left| u(\theta) - \sum_{\lambda=1}^{m} a_\lambda e^{i\lambda\theta} \right|^2 \right\} \right\} \\
&= \mathfrak{M}^-_\theta \left\{ \left| u(\theta) - \sum_{\lambda=1}^{m} a_\lambda e^{i\lambda\theta} \right|^2 \right\} \\
&= \left\| u^-(.) - \sum_{\lambda=1}^{m} a_\lambda e^{i\lambda(.)} \right\|^2_{B^2(\mathbb{R},\mathbb{R}^n)},
\end{aligned}$$

and by using the fact that $u^- \in \mathcal{B}$, which implies that

$$\lim_{m\to+\infty} \left\| u^-(.) - \sum_{\lambda=1}^{m} a_\lambda e^{i\lambda(.)} \right\|_{B^2(\mathbb{R},\mathbb{R}^n)} = 0,$$

we obtain

$$\lim_{m\to+\infty} \left\| \tilde{u}(.) - \sum_{\lambda=1}^{m} b_\lambda e^{i\lambda(.)} \right\|^2_{B^2(\mathbb{R},\mathcal{B})} = 0.$$

In the other hand it is very easy to see that $b_\lambda \in \mathcal{AP}^0((-\infty,0],\mathbb{R}^n) \subset \mathcal{B}$, and that

$$\sum_\lambda |b_\lambda|_{\mathcal{B}}^2 = \sum_\lambda \left(\mathfrak{M}_\theta^- \left\{\left|a_\lambda e^{i\lambda\theta}\right|^2\right\}\right) = \sum_\lambda |a_\lambda|^2 < \infty,$$

and so $\{b_\lambda\}_\lambda \in l^2(\mathcal{B})$. This implies that $\tilde{u} \in B^2(\mathbb{R},\mathcal{B})$, and $\tilde{u}(t) \sim \sum_\lambda b_\lambda e^{i\lambda t}$, and so (i) and (ii) are proven.

Applying the Parseval equality, (c.f. [4, page 109], [29, Proposition 2.8]), we obtain

$$\|\tilde{u}\|_{B^2(\mathbb{R},\mathcal{B})}^2 = \sum_\lambda \left|b_\lambda e^{i\lambda t}\right|_{\mathcal{B}}^2 = \sum_\lambda |a_\lambda|^2 := \|u\|_{B^2(\mathbb{R},\mathbb{R}^n)}^2,$$

and so (iii) is proven. □

Theorem 3.4. *Suppose $u \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$ is such that $u(t) \sim \sum_\lambda a_\lambda e^{i\lambda t}$. Then $\tilde{u} \in B^{1,2}(\mathbb{R},\mathcal{B})$, and $\nabla\tilde{u}(t) = \nabla(u_t) = (\nabla u)_t \sim \sum_\lambda i\lambda b_\lambda e^{i\lambda t}$, $\forall t \in \mathbb{R}$.*

Proof. $u \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$ implies that $u \in B^2(\mathbb{R},\mathbb{R}^n)$ and $\sum_\lambda |i\lambda a_\lambda|^2 < \infty$.
Hence, using Lemma (3.3) we have

$$\tilde{u} \in B^2(\mathbb{R},\mathcal{B}), \text{ and } \tilde{u}(t) \sim \sum_\lambda b_\lambda e^{i\lambda t}.$$

However $\sum_\lambda |i\lambda b_\lambda|_{\mathcal{B}}^2 = \sum_\lambda \left(\mathfrak{M}_\theta^- \left\{\left|i\lambda a_\lambda e^{i\lambda\theta}\right|^2\right\}\right) = \sum_\lambda |i\lambda a_\lambda|^2 < \infty$, and from (2.2), we obtain

$$\tilde{u} \in B^{1,2}(\mathbb{R},\mathcal{B}), \text{and } \nabla\tilde{u}(t) = \nabla(u_t) \sim \sum_\lambda i\lambda b_\lambda e^{i\lambda t}.$$

By virtue of Lemma (3.1), $\nabla u \in B^2(\mathbb{R},\mathbb{R}^n)$ implies that $[t \to (\nabla u)_t] \in B^2(\mathbb{R},\mathcal{B})$.
In the other hand, since the mean value is translation invariant, (c.f. [29, page 10]), we have for all t in $\mathbb{R}$

$$\begin{aligned}
\left\|(\nabla u)_{(.)} - \sum_{\lambda=1}^m i\lambda b_\lambda e^{i\lambda(.)}\right\|_{B^2(\mathbb{R},\mathcal{B})}^2 &= \mathfrak{M}_t\left\{\left|(\nabla u)_t - \sum_{\lambda=1}^m i\lambda b_\lambda e^{i\lambda t}\right|_{\mathcal{B}}^2\right\} \\
&= \mathfrak{M}_t\left\{\mathfrak{M}_\theta^-\left\{\left|\nabla u(t+\theta) - \sum_{\lambda=1}^m i\lambda a_\lambda e^{i\lambda(t+\theta)}\right|^2\right\}\right\} \\
&= \mathfrak{M}_t\left\{\mathfrak{M}_\theta^-\left\{\left|\nabla u(\theta) - \sum_{\lambda=1}^m i\lambda a_\lambda e^{i\lambda\theta}\right|^2\right\}\right\} \\
&= \mathfrak{M}_\theta^-\left\{\left|\nabla u(\theta) - \sum_{\lambda=1}^m i\lambda a_\lambda e^{i\lambda\theta}\right|^2\right\}.
\end{aligned}$$

Since $u^- \in B^{1,2}((-\infty,0],\mathbb{R}^n)$ if $u \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$, we obtain

$$\lim_{m\to+\infty} \mathfrak{M}_\theta^-\left\{\left|\nabla u(\theta) - \sum_{\lambda=1}^m i\lambda a_\lambda e^{i\lambda\theta}\right|^2\right\} = 0,$$

and so

$$\lim_{m\to+\infty}\left\|(\nabla u)_{(.)}-\sum_{\lambda=1}^{m} i\lambda b_\lambda e^{i\lambda(.)}\right\|^2_{B^{1,2}(\mathbb{R},\mathcal{B})}=0,$$

which implies that $(\nabla u)_t \sim \sum_\lambda i\lambda b_\lambda e^{i\lambda t}$, and by the unicity of Fourier-Bohr series [4, 25], we conclude that $\nabla(u_t)=(\nabla u)_t$. □

Lemma 3.5. *Suppose that* $[t_1\mapsto[t_2\mapsto \mathfrak{f}(t_1)(t_2)]]$ *and* $[t_1\mapsto[t_2\mapsto \mathfrak{g}(t_1)(t_2)]]$ *are in* $B^2(\mathbb{R},B^2(\mathbb{R},\mathbb{R}^n))$. *Then*

$$\mathfrak{M}_{t_1}\{\mathfrak{M}_{t_2}\{\mathfrak{f}(t_1)(t_2).\mathfrak{g}(t_1)(t_2)\}\}=\mathfrak{M}_{t_2}\{\mathfrak{M}_{t_1}\{\mathfrak{f}(t_1)(t_2).\mathfrak{g}(t_1)(t_2)\}\}.$$

Proof. Since for all $\mathbb{E}$ Banach space, $B^2(\mathbb{R},\mathbb{E})$ is isomorphic to $L^2(\mathbb{R}_{\mathbf{B}},\mathbb{E})$, where $L^2(\mathbb{R}_{\mathbf{B}},\mathbb{E})$ is taken with respect to the Haar measure $d\mu$ on the compact group $\mathbb{R}_{\mathbf{B}}$, (we recall that $\mathbb{R}_{\mathbf{B}}$ is the Bohr compactification of $\mathbb{R}$), (see [29, Chapter 1]), then there exist unique $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$ in $L^2(\mathbb{R}_{\mathbf{B}},L^2(\mathbb{R}_{\mathbf{B}},\mathbb{R}^n))$, where $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$ are the extensions of $\mathfrak{f}$ and $\mathfrak{g}$ to $\mathbb{R}_{\mathbf{B}}\times\mathbb{R}_{\mathbf{B}}$, and we have

$$\mathfrak{M}_{t_1}\{\mathfrak{M}_{t_2}\{\mathfrak{f}(t_1)(t_2).\mathfrak{g}(t_1)(t_2)\}\}=\int_{\mathbb{R}_{\mathbf{B}}}\left(\int_{\mathbb{R}_{\mathbf{B}}}\tilde{\mathfrak{f}}(r)(s).\tilde{\mathfrak{g}}(r)(s)d\mu(s)\right)d\mu(r).$$

It follows that $\left[(r,s)\mapsto\left|\tilde{\mathfrak{f}}(r)(s).\tilde{\mathfrak{g}}(r)(s)\right|\right]$ is a measurable-positive function. Since $\forall r\in\mathbb{R}_{\mathbf{B}}$, $\left[s\mapsto\tilde{\mathfrak{f}}(r)(s)\right]$, and $[s\mapsto\tilde{\mathfrak{g}}(r)(s)]$ are in $L^2(\mathbb{R}_{\mathbf{B}},\mathbb{R}^n)$, by using the Hölder inequality [14], we obtain $\left[s\mapsto\tilde{\mathfrak{f}}(r)(s).\tilde{\mathfrak{g}}(r)(s)\right]\in L^1(\mathbb{R}_{\mathbf{B}},\mathbb{R})$, and so

$$\forall r\in\mathbb{R}_{\mathbf{B}},\quad \left[r\mapsto\int_{\mathbb{R}_{\mathbf{B}}}\left|\tilde{\mathfrak{f}}(r)(s).\tilde{\mathfrak{g}}(r)(s)\right|d\mu(s)\right]<+\infty.$$

Since $L^2(\mathbb{R}_{\mathbf{B}},L^2(\mathbb{R}_{\mathbf{B}},\mathbb{R}^n))\subset L^2(\mathbb{R}_{\mathbf{B}},L^1(\mathbb{R}_{\mathbf{B}},\mathbb{R}^n))$, we have $\left[r\mapsto\left[s\mapsto\tilde{\mathfrak{f}}(r)(s)\right]\right]$ and $[r\mapsto[s\mapsto\tilde{\mathfrak{g}}(r)(s)]]$ in $L^2(\mathbb{R}_{\mathbf{B}},L^1(\mathbb{R}_{\mathbf{B}},\mathbb{R}^n))$, and by the Hölder inequality we obtain

$$\left[r\mapsto\left[s\mapsto\tilde{\mathfrak{f}}(r)(s).\tilde{\mathfrak{g}}(r)(s)\right]\right]\in L^1(\mathbb{R}_{\mathbf{B}},L^1(\mathbb{R}_{\mathbf{B}},\mathbb{R}^n)),$$

and so $\int_{\mathbb{R}_{\mathbf{B}}}\left(\int_{\mathbb{R}_{\mathbf{B}}}\left|\tilde{\mathfrak{f}}(r)(s).\tilde{\mathfrak{g}}(r)(s)\right|d\mu(s)\right)d\mu(r)<+\infty$, which implies by using Tonelli Theorem, (c.f. [14, page 54]), that $\left[(r,s)\mapsto\tilde{\mathfrak{f}}(r)(s).\tilde{\mathfrak{g}}(r)(s)\right]\in L^1(\mathbb{R}_{\mathbf{B}}\times\mathbb{R}_{\mathbf{B}},\mathbb{R}))$. And by using Fubini Theorem (c.f. [14, page 55]), we obtain

$$\int_{\mathbb{R}_{\mathbf{B}}}\left(\int_{\mathbb{R}_{\mathbf{B}}}\tilde{\mathfrak{f}}(r)(s).\tilde{\mathfrak{g}}(r)(s)d\mu(s)\right)d\mu(r)=\int_{\mathbb{R}_{\mathbf{B}}}\left(\int_{\mathbb{R}_{\mathbf{B}}}\tilde{\mathfrak{f}}(r)(s).\tilde{\mathfrak{g}}(r)(s)d\mu(r)\right)d\mu(s).$$

Consequently

$$\mathfrak{M}_{t_1}\{\mathfrak{M}_{t_2}\{\mathfrak{f}(t_1)(t_2).\mathfrak{g}(t_1)(t_2)\}\}=\mathfrak{M}_{t_2}\{\mathfrak{M}_{t_1}\{\mathfrak{f}(t_1)(t_2).\mathfrak{g}(t_1)(t_2)\}\}.$$

□

Let $\mathcal{T}$ be the operator defined by :

$$\begin{aligned}\mathcal{T}:\quad & B^2(\mathbb{R},\mathbb{R}^n)\to B^2(\mathbb{R},\mathcal{B})\\ & u\mapsto[t\mapsto u_t].\end{aligned}$$

$\mathcal{T}$ is a linear continuous operator between two Hilbert spaces, then its adjoint operator $\mathcal{T}^*$ is well defined in $B^2(\mathbb{R},\mathcal{B})^*$ into $B^2(\mathbb{R},\mathbb{R}^n)^*$, linear and continuous (cf. [1, Chapter 2]), and satisfies the following relation

$$\langle \check{\varphi} | \mathcal{T}(u)\rangle_{B^2(\mathbb{R},\mathcal{B})} = \langle \mathcal{T}^*(\check{\varphi})|u\rangle_{B^2(\mathbb{R},\mathbb{R}^n)},$$

for all $\check{\varphi} \in B^2(\mathbb{R},\mathcal{B})^*$ and $u \in B^2(\mathbb{R},\mathbb{R}^n)^* \equiv B^2(\mathbb{R},\mathbb{R}^n)$. In the following Lemma, we will specify the form of $\mathcal{T}^*$ on $B^2(\mathbb{R},\mathcal{B})$.

Lemma 3.6. *The adjoint operator of $\mathcal{T}$ has the following form in $B^2(\mathbb{R},\mathcal{B})$*

$$\begin{aligned} \mathcal{T}^* : \quad & B^2(\mathbb{R},\mathcal{B}) \longrightarrow B^2(\mathbb{R},\mathbb{R}^n) \\ & [t \mapsto [\theta \mapsto \varphi(t)(\theta)]] \mapsto \left[t \mapsto \mathfrak{M}_\theta^- \{\varphi(t-\theta)(\theta)\}\right]. \end{aligned}$$

Proof. Let $\check{\varphi} \in B^2(\mathbb{R},\mathcal{B})^*$. By Riesz-Frechet Theorem [14, page 81], there exists unique $\varphi \in B^2(\mathbb{R},\mathcal{B})$, such that for all $\psi \in B^2(\mathbb{R},\mathcal{B})$

$$\langle \check{\varphi}|\psi\rangle_{B^2(\mathbb{R},\mathcal{B})} = \mathfrak{M}_t \left\{(\varphi(t)|\psi(t))_{\mathcal{B}}\right\}.$$

And so, by using Remark (3.2), Lemma (3.5), and the invariance of the mean value by translation, we have $\forall u \in B^2(\mathbb{R},\mathbb{R}^n)$

$$\begin{aligned} \langle \check{\varphi}|\mathcal{T}(u)\rangle_{B^2(\mathbb{R},\mathcal{B})} &= \mathfrak{M}_t \left\{(\varphi(t)|\mathcal{T}(u)(t))_{\mathcal{B}}\right\} \\ &= \mathfrak{M}_t \left\{\mathfrak{M}_\theta^- \{\varphi(t)(\theta).u_t(\theta)\}\right\} \\ &= \mathfrak{M}_t \left\{\mathfrak{M}_\theta^- \{\varphi(t)(\theta).u(t+\theta)\}\right\} \\ &= \mathfrak{M}_t \left\{\mathfrak{M}_{t'} \left\{\varphi(t)(t').u(t+t')\right\}\right\} \\ &= \mathfrak{M}_{t'} \left\{\mathfrak{M}_t \left\{\varphi(t)(t').u(t+t')\right\}\right\} \\ &= \mathfrak{M}_{t'} \left\{\mathfrak{M}_t \left\{\varphi(t-t')(t').u(t)\right\}\right\} \\ &= \mathfrak{M}_t \left\{\mathfrak{M}_{t'} \left\{\varphi(t-t')(t').u(t)\right\}\right\} \\ &= \mathfrak{M}_t \left\{\mathfrak{M}_{t'} \left\{\varphi(t-t')(t')\right\}.u(t)\right\} \\ &= \mathfrak{M}_t \left\{\mathfrak{M}_\theta^- \{\varphi(t-\theta)(\theta)\}.u(t)\right\} \\ &= \left\langle \mathfrak{M}_\theta^- \{\varphi(t-\theta)(\theta)\}|u(t)\right\rangle_{B^2(\mathbb{R},\mathbb{R}^n)} \\ &= \langle \mathcal{T}^*(\varphi)|u\rangle_{B^2(\mathbb{R},\mathbb{R}^n)}. \end{aligned}$$

And so $\forall \varphi \in B^2(\mathbb{R},\mathcal{B})$, $\mathcal{T}^*(\varphi)(t) = \mathfrak{M}_\theta^- \{\varphi(t-\theta)(\theta)\}$. □

4 Euler Lagrange Equation (Variational Principle)

Let $L : \mathbb{H} \times \mathbb{R} \to \mathbb{R}$, $(X,t) \to L(X,t)$ be a continuous function.
We give the following list of assumptions :

(H1) $L \in \mathcal{APU}(\mathbb{H}\times\mathbb{R},\mathbb{R})$, such that there exists $\alpha \in (0,+\infty)$, and $a_1 \in [0,+\infty)$; $\forall t \in \mathbb{R}$, $\forall X,Y \in \mathbb{H}$, $|L(X,t) - L(Y,t)| < a_1 |X-Y|^\alpha_{\mathbb{H}}$.

(H2) $L \in \mathcal{APU}(\mathbb{H}\times\mathbb{R},\mathbb{R})$, such that the partial differential with respect to $X \in \mathbb{H}$, $L_X(X,t)$ exists for all $(X,t) \in \mathbb{H}\times\mathbb{R}$, and $L_X \in \mathcal{APU}(\mathbb{H}\times\mathbb{R},\mathcal{L}(\mathbb{H},\mathbb{R}))$.

(H3) There exists $a_2 \in [0,+\infty)$ such that $|L_X(X,t) - L_X(Y,t)| < a_2 |X-Y|_{\mathbb{H}}$ for all $X,Y \in \mathbb{H}$, for all $t \in \mathbb{R}$.

(H4) $L \in \mathcal{APU}(\mathbb{H}\times\mathbb{R},\mathbb{R})$, the partial differential $D_kL(x_1,\varphi_1,x_2,\varphi_2,t)$ exists for all $(x_1,\varphi_1,x_2,\varphi_2,t) \in \mathbb{H}\times\mathbb{R}$, $D_kL \in \mathcal{APU}(\mathbb{H}\times\mathbb{R},\mathcal{L}(\mathbb{R}^n,\mathbb{R}))$ for $k \in \{1,3\}$, and $D_kL \in \mathcal{APU}(\mathbb{H}\times\mathbb{R},\mathcal{L}(\mathcal{B},\mathbb{R}))$ for $k \in \{2,4\}$.

Lemma 4.1. *We assume condition (H1) fulfilled. Let $p,q \in [1,+\infty)$ be such that $\frac{p}{q} = \alpha$, then the following two assertions hold:*

(i) If $\kappa \in B^p(\mathbb{R},\mathbb{H})$, then $[t \mapsto L(\kappa(t),t)] \in B^q(\mathbb{R},\mathbb{R})$.

(ii) The Nemytskii operator on L, $\mathcal{N}_L : B^p(\mathbb{R},\mathbb{H}) \to B^q(\mathbb{R},\mathbb{R})$ defined by

$$\mathcal{N}_L(\kappa)(t) := L(\kappa(t),t),$$

satisfies $\left\|\mathcal{N}_L(\kappa_1) - \mathcal{N}_L(\kappa_2)\right\|_{B^q(\mathbb{R},\mathbb{R})} \leq a_1 \left\|\kappa_1 - \kappa_2\right\|^{\alpha}_{B^p(\mathbb{R},\mathbb{H})}$.

Proof. Setting $g(t) := L(0,t)$, and so $g \in \mathcal{AP}^0(\mathbb{R},\mathbb{R})$ which implies that $g \in L^q_{loc}(\mathbb{R},\mathbb{R})$ (the Lebesgue space). The Hölder assumption (H1) implies

$$|L(X,t)| \leq a_1 |X|^{\alpha}_{\mathbb{H}} + g(t), \quad \forall (X,t) \in \mathbb{H}\times\mathbb{R}.$$

By using [24, Chapter 1], $\kappa \in L^q_{loc}(\mathbb{R},\mathbb{H})$ implies $[t \mapsto L(\kappa(t),t)] \in L^q_{loc}(\mathbb{R},\mathbb{R})$.

Since $\kappa \in B^p(\mathbb{R},\mathbb{H})$, there exists $\{\kappa_j\}_j \in \mathcal{AP}^0(\mathbb{R},\mathbb{H})$ such that

$$\lim_{j\to+\infty} \left\|\kappa - \kappa_j\right\|_{B^p(\mathbb{R},\mathbb{H})} = 0.$$

By using [32, Theorem 2.7, page 16], setting $\chi_j(t) := L(\kappa_j(t),t)$ we have $\chi_j \in \mathcal{AP}^0(\mathbb{R},\mathbb{R})$, and a straightforward calculation gives us the following inequality :

$$\overline{\mathfrak{M}}_t\left\{\left|L(\kappa(t),t) - \chi_j(t)\right|^q\right\}^{\frac{1}{q}} \leq a_1 \overline{\mathfrak{M}}_t\left\{\left|\kappa(t) - \kappa_j(t)\right|^q_{\mathbb{H}}\right\}^{\frac{1}{q}} = a_1 \left\|\kappa - \kappa_j\right\|^{\alpha}_{B^p(\mathbb{R},\mathbb{H})},$$

and consequently we obtain

$$\lim_{j\mapsto+\infty} \overline{\mathfrak{M}}_t\left\{\left|L(\kappa(t),t) - \chi_j(t)\right|^q\right\}^{\frac{1}{q}} = 0,$$

that implies $[t \mapsto L(\kappa(t),t)] \in B^q(\mathbb{R},\mathbb{R})$, and so (i) is proven.
Moreover the last inequality becomes the one of (ii), when we replace $\kappa(t)$ by $\kappa_1(t)$ and $\chi_j(t)$ by $L(\kappa_2(t),t)$. □

Lemma 4.2. *We assume conditions (H2) and (H3) fulfilled. Then the Nemytskii operator $\mathcal{N}_L : B^2(\mathbb{R},\mathbb{H}) \to B^1(\mathbb{R},\mathbb{R})$, defined by $\mathcal{N}_L(\kappa)(t) := L(\kappa(t),t)$, is of class $\mathcal{C}^1$, and for all $\kappa, \delta\kappa \in B^2(\mathbb{R},\mathbb{H})$,*

$$(D\mathcal{N}_L(\kappa)\delta\kappa)(t) = L_X(\kappa(t),t).\delta\kappa(t).$$

Proof. • We show that there exist $c_2 \in [0,+\infty)$, $h \in B^1(\mathbb{R},\mathbb{H})$, such that for all $X \in \mathbb{H}$ and $t \in \mathbb{R}$ $|L(X,t)| \leq c_2 |X|_{\mathbb{H}}^2 + h(t)$.
By using (H3) for $Y = 0$, we obtain

$$\exists a_2 \in [0,\infty); \quad |L_X(X,t) - L_X(0,t)| \leq a_2 |X|_{\mathbb{H}},$$

which implies that

$$|L_X(X,t)| \leq a_2 |X|_{\mathbb{H}} + |L_X(0,t)|,$$

and so the mean value theorem [2, page 144], gives that, for all $(X,t) \in \mathbb{H} \times \mathbb{R}$

$$\begin{aligned}
|L(X,t)| &\leq |L(X,t) - L(0,t)| + |L(0,t)| \\
&\leq \sup_{H \in]0,X[} |L_X(H,t)| \, |X - 0|_{\mathbb{H}} + |L(0,t)| \\
&\leq (a_2 |X|_{\mathbb{H}} + |L_X(0,t)|) \, |X|_{\mathbb{H}} + |L(0,t)| \\
&= a_2 |X|_{\mathbb{H}}^2 + |L_X(0,t)| . |X|_{\mathbb{H}} + |L(0,t)| \\
&\leq a_2 |X|_{\mathbb{H}}^2 + \frac{1}{2} |L_X(0,t)|^2 + \frac{1}{2} |X|_{\mathbb{H}}^2 + |L(0,t)| \\
&= \left(a_2 + \frac{1}{2}\right) |X|_{\mathbb{H}}^2 + \frac{1}{2} |L_X(0,t)|^2 + |L(0,t)|,
\end{aligned}$$

and by setting $h(t) := \frac{1}{2} |L_X(0,t)|^2 + |L(0,t)|$, and $c_1 := \left(a_2 + \frac{1}{2}\right)$, since $L \in \mathcal{APU}(\mathbb{H} \times \mathbb{R},\mathbb{R})$, and $L_X \in \mathcal{APU}(\mathbb{H} \times \mathbb{R}, \mathcal{L}(\mathbb{H},\mathbb{R}))$, we obtain $h \in \mathcal{AP}^0(\mathbb{R},\mathbb{H}) \subset B^1(\mathbb{R},\mathbb{H})$.

• We show that if $\kappa \in B^2(\mathbb{R},\mathbb{H})$, then $[t \to L(\kappa(t),t)] \in B^1(\mathbb{R},\mathbb{R})$.
Let $\kappa \in B^2(\mathbb{R},\mathbb{H})$, then $|L(\kappa(t),t)| \leq c_2 |\kappa(t)|_{\mathbb{H}}^2 + h(t)$, where $h \in B^1(\mathbb{R},\mathbb{H}) \subset L^1_{loc}(\mathbb{R},\mathbb{H})$, implies that

$$[t \to L(\kappa(t),t)] \in L^1_{loc}(\mathbb{R},\mathbb{R}).$$

Applying Lemma (4.1) to the function L_X with $p = 2, q = 2, \alpha = 1$, we have

$$[t \to L_X(\kappa(t),t)] \in B^2(\mathbb{R}, \mathcal{L}(\mathbb{H},\mathbb{R})).$$

Let $\{\kappa_j\}_j$ be a sequence in $\mathcal{AP}^0(\mathbb{R},\mathbb{H})$ such that

$$\lim_{j \to \infty} \left\| \kappa - \kappa_j \right\|_{B^2(\mathbb{R},\mathbb{H})} = 0.$$

Using the mean value inequality theorem [2, page 144], we obtain for all $t \in \mathbb{R}$

$$\begin{aligned}
|L(\kappa(t),t) - L(\kappa_j(t),t) - &L_X(\kappa(t),t).(\kappa(t) - \kappa_j(t))| \\
&\leq \sup_{\zeta \in]\kappa(t),\kappa_j(t)[} |L_X(\zeta,t) - L_X(\kappa(t),t)| \left|\kappa - \kappa_j\right|_{\mathbb{H}} \\
&\leq a_2 \sup_{\zeta \in]\kappa(t),\kappa_j(t)[} |\zeta - \kappa(t)|_{\mathbb{H}} \left|\kappa - \kappa_j\right|_{\mathbb{H}} \\
&\leq a_2 \left|\kappa(t) - \kappa_j(t)\right|_{\mathbb{H}}^2,
\end{aligned}$$

and by the monoticity of the mean value we obtain :

$$\overline{\mathfrak{M}}_t \left\{ \left| L(\kappa(t),t) - L(\kappa_j(t),t) - L_X(\kappa(t),t).(\kappa(t) - \kappa_j(t)) \right| \right\} \leq a_2 \overline{\mathfrak{M}}_t \left\{ \left|\kappa(t) - \kappa_j(t)\right|_{\mathbb{H}}^2 \right\}.$$

Since $[t \to L_X(\kappa(t),t)] \in B^2(\mathbb{R},\mathcal{L}(\mathbb{H},\mathbb{R}))$, and $\kappa-\kappa_j \in B^2(\mathbb{R},\mathbb{H})$, we have

$$[t \to L_X\,(\kappa(t),t)\,(\kappa(t)-\kappa_j\,(t))] \in B^1(\mathbb{R},\mathbb{R}).$$

Thanks to [32, Theorem 2.7, page 16], we have

$$[t \to L(\kappa_j(t),t)] \in \mathcal{AP}^0(\mathbb{R},\mathbb{R}) \subset B^1(\mathbb{R},\mathbb{R}),$$

and so by setting $\phi_j(t) := L(\kappa_j(t),t) - L_X(\kappa(t),t)(\kappa(t)-\kappa_j(t))$, we have $\phi_j \in B^1(\mathbb{R},\mathbb{R})$. The last previous inequality implies

$$\lim_{j\to\infty} \overline{\mathfrak{M}}_t\left\{\left|L(\kappa(t),t)-\phi_j(t)\right|\right\} = 0.$$

Consequently

$$[t \to L(\kappa(t),t)] \in B^1(\mathbb{R},\mathbb{R}).$$

• We show that for all $\kappa \in B^2(\mathbb{R},\mathbb{H})$ the operator

$$\mathcal{L}(\kappa) : B^2(\mathbb{R},\mathbb{H}) \to B^1(\mathbb{R},\mathbb{R}), \quad (\mathcal{L}(\kappa)\delta\kappa)(t) := L_X(\kappa(t),t)\delta\kappa(t)$$

is linear continuous.
We have already seen that $[t \to L_X(\kappa(t),t)] \in B^1(\mathbb{R},\mathbb{R})$. The linearity of $\mathcal{L}(\kappa)$ is easy to verify, and by using a Cauchy-Schwartz-Buniakovski inequality, [4, page 69], we have

$$\begin{aligned}\mathfrak{M}_t\{|L_X(\kappa(t),t).\delta\kappa(t)|\} &\leq \mathfrak{M}_t\{|L_X(\kappa(t),t)||\delta\kappa(t)|_{\mathbb{H}}\}\\ &\leq \mathfrak{M}_t\left\{|L_X(\kappa(t),t)|^2\right\}^{\frac{1}{2}}\mathfrak{M}_t\left\{|\delta\kappa(t)|^2_{\mathbb{H}}\right\}^{\frac{1}{2}},\end{aligned}$$

and so $\mathcal{L}(\kappa)$ is continuous.

• We show the differentiability of $\mathcal{N}_L$.
Let $\kappa \in B^2(\mathbb{R},\mathbb{H})$. By using the mean value inequality theorem, [2, page 144], we have for all $t \in \mathbb{R}$,

$$\begin{aligned}&|L(\kappa(t)+\delta\kappa(t),t) - L(\kappa(t),t) - L_X(\kappa(t),t).\delta\kappa(t)|\\ &\qquad\leq \sup_{\zeta\in]\kappa(t),\kappa(t)+\delta\kappa(t)[} |L_X(\zeta(t),t) - L_X(\kappa(t),t)|.\,|\delta\kappa(t)|_{\mathbb{H}}\\ &\qquad\leq a_2\,|\delta\kappa(t)|^2_{\mathbb{H}}.\end{aligned}$$

And the monotonicity of the mean value permits us to obtain

$$\mathfrak{M}_t\{|L(\kappa(t)+\delta\kappa(t),t) - L(\kappa(t),t) - L_X(\kappa(t),t).\delta\kappa(t)|\} \leq a_2\mathfrak{M}_t\left\{|\delta\kappa(t)|^2_{\mathbb{H}}\right\},$$

which provides

$$\left\|\mathcal{N}_L(\kappa+\delta\kappa) - \mathcal{N}_L(\kappa) - \mathcal{L}(\kappa).\delta\kappa\right\|_{B^1(\mathbb{R},\mathbb{R})} \leq a_2\,\|\delta\kappa\|^2_{B^2(\mathbb{R},\mathbb{H})},$$

and so that $\mathcal{N}_L$ is differentiable at κ and $D\mathcal{N}_L(\kappa) = \mathcal{L}(\kappa)$.

• We show that $\mathcal{N}_L$ is of class $\mathcal{C}^1$.
Let $\kappa_1, \kappa_2 \in B^2(\mathbb{R}, \mathbb{H})$. From (H3), we have for all $\delta\kappa \in B^2(\mathbb{R}, \mathbb{H})$ such that $\|\delta\kappa\|_{B^2(\mathbb{R},\mathbb{H})} \leq 1$, for all $t \in \mathbb{R}$:

$$\begin{aligned} |(L_X(\kappa_1(t),t) - L_X(\kappa_2(t),t))\delta\kappa(t)| &\leq |(L_X(\kappa_1(t),t) - L_X(\kappa_2(t),t))| \,.\, |\delta\kappa(t)|_{\mathbb{H}} \\ &\leq a_2 |\kappa_1(t) - \kappa_2(t)|_{\mathbb{H}} \,.\, |\delta\kappa(t)|_{\mathbb{H}}\,, \end{aligned}$$

that implies by using the Cauchy-Schwarz-Buniakovski inequality, [4, page 69]

$$\begin{aligned} \mathfrak{M}_t\{|(L_X(\kappa_1(t),t) - L_X(\kappa_2(t),t))\delta\kappa(t)|\} &\leq a_2\mathfrak{M}_t\{|\kappa_1(t) - \kappa_2(t)|_{\mathbb{H}} \,.\, |\delta\kappa(t)|_{\mathbb{H}}\} \\ &\leq a_2 \|\kappa_1 - \kappa_2\|_{B^2(\mathbb{R},\mathbb{H})} \,.\, \|\delta\kappa\|_{B^2(\mathbb{R},\mathbb{H})} \\ &\leq a_2 \|\kappa_1 - \kappa_2\|_{B^2(\mathbb{R},\mathbb{H})}\,. \end{aligned}$$

Therefore we have

$$\left\|D\mathcal{N}_L(\kappa_1) - D\mathcal{N}_L(\kappa_2)\right\|_{\mathcal{L}} \leq a_2 \|\kappa_1 - \kappa_2\|_{B^2(\mathbb{R},\mathbb{H})}\,,$$

which implies the continuity of $D\mathcal{N}_L$. ☐

Theorem 4.3 (Variational Principle)**.** *We assume conditions (H3) and (H4) fulfilled. Then the functional* $\Phi : B^{1,2}(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}$*, defined by*

$$\Phi(u) := \mathfrak{M}_t\{L(u(t), u_t, \nabla u(t), (\nabla u)_t, t)\}$$

is of class $\mathcal{C}^1$*, and the two following assertions are equivalent:*

1. $D\Phi(u) = 0$*, i.e.* u *is a critical point of* Φ*.*
2. $D_1L(u(t), u_t, \nabla u(t), (\nabla u)_t, t) + \mathcal{T}^* D_2L(u(t), u_t, \nabla u(t), (\nabla u)_t, t)$
$= \nabla\left[D_3L(u(t), u_t, \nabla u(t), (\nabla u)_t, t) + \mathcal{T}^* D_4L(u(t), u_t, \nabla u(t), (\nabla u)_t, t)\right]$

Proof. We consider the operator $\mathfrak{L} : B^{1,2}(\mathbb{R}, \mathbb{R}^n) \to B^2(\mathbb{R}, \mathbb{R}^n) \times B^2(\mathbb{R}, \mathcal{B}) \times B^2(\mathbb{R}, \mathbb{R}^n) \times B^2(\mathbb{R}, \mathcal{B}) \equiv B^2(\mathbb{R}, \mathbb{R}^n \times \mathcal{B} \times \mathbb{R}^n \times \mathcal{B}) := B^2(\mathbb{R}, \mathbb{H})$ defined by

$$\mathfrak{L}(u)(t) := (u(t), u_t, \nabla u(t), (\nabla u)_t).$$

It follows that $\mathfrak{L}$ is linear continuous, therefore $\mathfrak{L}$ is of class $\mathcal{C}^1$, and $D\mathfrak{L}(u)v = \mathfrak{L}(v)$.

By using Lemma (4.2), the Nemytskii operator $\mathcal{N}_L : B^2(\mathbb{R}, \mathbb{H}) \to B^1(\mathbb{R}, \mathbb{R})$, defined by $\mathcal{N}_L(\kappa)(t) := L(\kappa(t), t)$, is of class $\mathcal{C}^1$ and for all $\kappa, \delta\kappa \in B^2(\mathbb{R}, \mathbb{H})$ we have

$$(D\mathcal{N}_L(\kappa)\delta\kappa)(t) = L_X(\kappa(t), t)\delta\kappa(t).$$

It follows that the mean value $\mathfrak{M} : B^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ is linear continuous, therefore it is of class $\mathcal{C}^1$, and $D\mathfrak{M}\{f\}.g = \mathfrak{M}\{g\}$ for all $f, g \in B^1(\mathbb{R}, \mathbb{R})$.

Consequently $\Phi : \mathfrak{M} \circ \mathcal{N}_L \circ \mathfrak{L}$ is of class $\mathcal{C}^1$ as a composition of three mappings of class $\mathcal{C}^1$, and by using the chain rule, we obtain, for all $u, v \in B^{1,2}(\mathbb{R}, \mathbb{R}^n)$, the following formula

$$\begin{aligned} D\Phi(u).v &= D\mathfrak{M}(\mathcal{N}_L \circ \mathfrak{L}(u)) \circ D\mathcal{N}_L(\mathfrak{L}(u)) \circ D\mathfrak{L}(u)v \\ &= \mathfrak{M}\left\{D\mathcal{N}_L(\mathfrak{L}(u)).\mathfrak{L}(v)\right\} \\ &= \mathfrak{M}_t\{D_1L(\underline{u}(t), t)v(t) + D_2L(\underline{u}(t), t)v_t \\ &+ D_3L(\underline{u}(t), t)\nabla v(t) + D_4L(\underline{u}(t), t)(\nabla v)_t\}. \end{aligned}$$

Let $u \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$, we assume that (i) is true. Then for all $v \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$ we have

$$\begin{aligned}
0 &= D\Phi(u).v \\
&= \mathfrak{M}_t\{D_1L(\underline{u}(t),t)v(t) + D_2L(\underline{u}(t),t)v_t \\
&\quad + D_3L(\underline{u}(t),t)\nabla v(t) + D_4L(\underline{u}(t),t)(\nabla v)_t\} \\
&= \mathfrak{M}_t\{D_1L(\underline{u}(t),t)v(t) + D_2L(\underline{u}(t),t)\mathcal{T}v(t) \\
&\quad + D_3L(\underline{u}(t),t)\nabla v(t) + D_4L(\underline{u}(t),t)\mathcal{T}(\nabla v)(t)\} \\
&= \mathfrak{M}_t\{D_1L(\underline{u}(t),t)v(t) + \mathcal{T}^*D_2L(\underline{u}(t),t)v(t) \\
&\quad + D_3L(\underline{u}(t),t)\nabla v(t) + \mathcal{T}^*D_4L(\underline{u}(t),t)\nabla v(t)\} \\
&= \mathfrak{M}_t\{(D_1L(\underline{u}(t),t) + \mathcal{T}^*D_2L(\underline{u}(t),t))v(t) \\
&\quad + (D_3L(\underline{u}(t),t) + \mathcal{T}^*D_4L(\underline{u}(t),t))\nabla v(t)\}.
\end{aligned}$$

Then by using [11, Proposition 10], we obtain (ii).

Conversly, if (ii) is true, then

$$[t \mapsto D_3L(\underline{u}(t),t) + \mathcal{T}^*D_4L(\underline{u}(t),t)] \in B^{1,2}(\mathbb{R},\mathbb{R}^n),$$

and for all $v \in \mathcal{AP}^1(\mathbb{R},\mathbb{R}^n)$ we have

$$\begin{aligned}
&\mathfrak{M}_t\{(D_1L(\underline{u}(t),t) + \mathcal{T}^*D_2L(\underline{u}(t),t)).v(t)\} \\
&\qquad\qquad - \mathfrak{M}_t\{\nabla(D_3L(\underline{u}(t),t) + \mathcal{T}^*D_4L(\underline{u}(t),t)).v(t)\} = 0.
\end{aligned}$$

Using [11, Propsition 9], we obtain

$$\begin{aligned}
0 &= \mathfrak{M}_t\{(D_1L(\underline{u}(t),t) + \mathcal{T}^*D_2L(\underline{u}(t),t)).v(t) \\
&\quad + (D_3L(\underline{u}(t),t) + \mathcal{T}^*D_4L(\underline{u}(t),t)).v'(t)\} \\
&= \mathfrak{M}_t\{D_1L(\underline{u}(t),t).v(t) + \mathcal{T}^*D_2L(\underline{u}(t),t).v(t) \\
&\quad + D_3L(\underline{u}(t),t).v'(t) + \mathcal{T}^*D_4L(\underline{u}(t),t).v'(t)\} \\
&= \mathfrak{M}_t\{D_1L(\underline{u}(t),t).v(t) + D_2L(\underline{u}(t),t)\mathcal{T}v(t) \\
&\quad + D_3L(\underline{u}(t),t).v'(t) + D_4L(\underline{u}(t),t)\mathcal{T}v'(t)\} \\
&= \mathfrak{M}_t\{D_1L(\underline{u}(t),t).v(t) + D_2L(\underline{u}(t),t)v_t \\
&\quad + D_3L(\underline{u}(t),t).v'(t) + D_4L(\underline{u}(t),t)v'_t\} \\
&= D\Phi(u).v.
\end{aligned}$$

Since $\mathcal{AP}^1(\mathbb{R},\mathbb{R}^n)$ is dense in $B^{1,2}(\mathbb{R},\mathbb{R}^n)$, [11, Proposition 8], we have $D\Phi(u).v = 0$, for all $v \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$, which implies $D\Phi(u) = 0$. $\square$

Definition 4.4. When $u \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$ satisfies the equation of (2) in Theorem (4.3), we say that u is a weak Besicovitch-ap solution of (1.2).

5 Existence and uniqueness

Let $L : \mathbb{H} \times \mathbb{R} \to \mathbb{R}$, $(X,t) \mapsto L(X,t)$ be a continuous function. We give the following list of assumptions.

(H5) $L(.,t)$ is convex, for all $t \in \mathbb{R}$.

(H6) There exists $a_3 \in [0,+\infty)$ such that for all $t \in \mathbb{R}$ and $(x_1,\varphi_1,x_2,\varphi_2) \in \mathbb{H}$, we have $|L(x_1,\varphi_1,x_2,\varphi_2,t)| \geq a_3(\beta+\gamma)$, where

$$\begin{cases} \beta := |x_1|^2 \text{ or } |\varphi_1|^2_{\mathcal{B}}, \\ \gamma := |x_2|^2 \text{ or } |\varphi_2|^2_{\mathcal{B}}. \end{cases}$$

Theorem 5.1 (Existence)**.** *We assume conditions (H3), (H4), (H5), and (H6) fulfilled. Then there exists a function $u \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$ which is a weak Besicovitch a.p. solution of* (1.2).

Proof. By using Theorem (4.3), the assumptions (H3) and (H4) imply that the functional Φ is of class $\mathcal{C}^1$. The assumption (H5) implies that the functional Φ is convex. And by using (iii) of Lemma (3.3) and the monoticity of the mean value, the assumption (H6) implies, for all $u \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$

$$\begin{aligned} \Phi(u) &\geq a_3\left(\|u\|^2_{B^2(\mathbb{R},\mathbb{R}^n)} + \|\nabla u\|^2_{B^2(\mathbb{R},\mathbb{R}^n)}\right) \\ &= a_3\,\|u\|^2_{B^{1,2}(\mathbb{R},\mathbb{R}^n)}, \end{aligned}$$

which allows us to deduce that Φ is coercive functional on $B^{1,2}(\mathbb{R},\mathbb{R}^n)$. And so by using [14, page 46], there exists $u \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$, such that

$$\Phi(u) = \inf\left\{\Phi(v); v \in B^{1,2}(\mathbb{R},\mathbb{R}^n)\right\}.$$

Therefore $D\Phi(u) = 0$, and applying Theorem (4.3), u is weak Besicovitch a.p. solution of (1.2) □

Theorem 5.2 (Uniqueness)**.** *Assuming that the conditions (H3), (H4), (H5), and (H6) are satisfied. Assume moreover that the following condition is satisfied,*

$$\begin{cases} \textit{There exists } a_4 \in [0,+\infty), \textit{ such that the function} \\ K : \mathbb{H}\times\mathbb{R} \to \mathbb{R} \textit{ defined by} \\ K(x_1,\varphi_1,x_2,\varphi_2,t) := L(x_1,\varphi_1,x_2,\varphi_2,t) - \frac{a_4}{2}(\beta+\gamma), \\ \textit{is convex with respect to } (x_1,\varphi_1,x_2,\varphi_2) \textit{ for all } t \in \mathbb{R}, \end{cases} \tag{5.1}$$

then there exists a unique function $u \in B^{1,2}(\mathbb{R},\mathbb{R}^n)$ which is a weak Besicovitch a.p. solution of (1.2).

Proof. By using the Theorem (5.1), the assumptions (H3), (H4), (H5), and (H6) ensure the existence of a weak Besicovitch a.p. solution of (1.2).

Setting the functional

$$\begin{aligned} \Psi(u) &:= \Phi(u) - \frac{a_4}{2}\left(\mathfrak{M}_t\left\{|u(t)|^2\right\} + \mathfrak{M}_t\left\{|\nabla u(t)|^2\right\}\right) \\ &= \Phi(u) - \frac{a_4}{2}\left(\|u\|^2_{B^2(\mathbb{R},\mathbb{R}^n)} + \|\nabla u\|^2_{B^2(\mathbb{R},\mathbb{R}^n)}\right). \end{aligned}$$

By (3i) of Lemma (3.3), the condition (5.1) implies that the functional Ψ is convex, and since Φ is of class $\mathcal{C}^1$, Ψ it is also, and for all $u, v \in B^{1,2}(\mathbb{R}, \mathbb{R}^n)$

$$\begin{aligned} D\Psi(u)v &= D\Phi(u)v - a_4 \left\{ \langle u, v \rangle_{B^2(\mathbb{R},\mathbb{R}^n)} + \langle \nabla u, \nabla v \rangle_{B^2(\mathbb{R},\mathbb{R}^n)} \right\} \\ &= D\Phi(u)v - a_4 \langle u, v \rangle_{B^{1,2}(\mathbb{R},\mathbb{R}^n)}. \end{aligned}$$

The Minty monotinicity of the differential of a convex functional, implies that

$$\langle D\Psi(u) - D\Psi(v), u - v \rangle_{B^{1,2}(\mathbb{R},\mathbb{R}^n)} \geq 0, \quad \forall u, v \in B^{1,2}(\mathbb{R}, \mathbb{R}^n),$$

and then we have $\forall u, v \in B^{1,2}(\mathbb{R}, \mathbb{R}^n)$

$$\langle D\Phi(u) - D\Phi(v), u - v \rangle_{B^{1,2}(\mathbb{R},\mathbb{R}^n)} - a_4 \langle u - v, u - v \rangle_{B^{1,2}(\mathbb{R},\mathbb{R}^n)} \geq 0,$$

eventually,

$$\langle D\Phi(u) - D\Phi(v), u - v \rangle_{B^{1,2}(\mathbb{R},\mathbb{R}^n)} \geq a_4 \|u - v\|^2_{B^{1,2}(\mathbb{R},\mathbb{R}^n)}, \quad \forall u, v \in B^{1,2}(\mathbb{R}, \mathbb{R}^n).$$

So if u and v are two weak Besicovitch a.p. solutions of (1.2), by Theorem (4.3) we have $D\Phi(u) = D\Phi(v) = 0$, and consequently $a_4 \|u - v\|^2_{B^{1,2}(\mathbb{R},\mathbb{R}^n)} = 0$, which gives that $u = v$. □

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